Mathematical formulation of mode-to-mode energy transfers and energy fluxes in compressible turbulence

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Understanding compressible turbulence is critical for modeling atmospheric, astrophysical, and engineering flows. However, compressible turbulence poses a more significant challenge than incompressible turbulence. We present a novel mathematical framework to compute mode-to-mode energy transfer rates and energy fluxes for compressible flows. The formalism captures detailed energy conservation within triads and allows decomposition of transfers into rotational, compressive, and mixed components, providing a clear picture of energy exchange among velocity and internal energy modes. We also establish analogies with incompressible hydrodynamic and magnetohydrodynamic flows, highlighting the framework's universality in studying energy transfers.

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I. INTRODUCTION

Compressible turbulence [1] is a fundamental phenomenon encountered in astrophysical applications [2], such as solar convection [3], astrophysical jets, and supernova explosions; in terrestrial atmospheres [4]; and in engineering applications, including fusion plasmas [5] and high-speed aerodynamics [6]. In a fluid flow, the relative density fluctuation $(\delta\rho)/\rho$ varies as $(U/C_s)^2$, where U is the velocity scale and C_s is the sound speed. For example, a river flow is incompressible because $U \ll C_s$. However, fluid flows in combustion and astrophysical jets are compressible with $U \gtrapprox C_s$. Compressible flows involve complex interactions and multiscale energy transfers among the solenoidal (or rotational) and compressive components of the velocity field, and internal energy. This paper constructs a mathematical framework to quantify triadic energy transfers and energy fluxes.

Compared with compressible turbulence, incompressible turbulence theory is well-established [7]. A key result in incompressible turbulence is Kolmogorov's spectrum $E(k) = K_{\mathrm{Ko}}\epsilon_{\mathrm{Ko}}^{2/3}k^{-5/3}$. This describes the kinetic-energy distribution among wavenumbers k in the inertial range, where ϵ_{Ko} is the constant energy flux and viscous dissipation rate, and K_{Ko} is Kolmogorov's constant [7–9]. Beyond the energy spectrum, the dynamics of scale interactions have also been explored in detail. Kraichnan [10] provided a framework of *combined energy transfers* among the velocity Fourier modes in a wavenumber triad. Later, Dar *et al.* [11] and Verma [12] generalized Kraichnan's

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[10] framework to mode-to-mode energy transfers. Moreover, energy is predominantly exchanged between neighboring wavenumber shells—a property known as *locality* of interactions—which underpins the scale-by-scale energy cascade in incompressible turbulence [13–19]. In contrast, the theoretical understanding of compressible turbulence is far less developed. The presence of an additional compressive velocity component and the coupling between the kinetic energy and internal energy via pressure-dilatation effects introduce significant complexity. As a result, the nature of interscale energy transfers, the role of compressibility, and the scaling laws remain active areas of research in compressible turbulence.

In the following discussion, we review past works on the energy spectra and transfers in compressible turbulence. Using numerical simulations, several researchers [20–24] showed that the rotational velocity component exhibits Kolmogorov's $k^{-5/3}$ energy spectrum, whereas the compressive component exhibits k^{-2} energy spectrum. The relative magnitude of compressive and rotational kinetic energies depends on the turbulent Mach number $M_t = U/C_s$, where U is the large-scale velocity, and C_s is the sound speed [20,21,25,26]. For example, using numerical simulations, Kida and Orszag [20] showed that an increase of M_t leads to a relative increase in compressive energy. Kida and Orszag [20,21], Miura and Kida [27], and Praturi and Girimaji [28] simulated compressible turbulence and computed the integrated energy transfers between kinetic and internal energies. Miura and Kida [27] investigated the energy exchange between kinetic and internal energies and identified pressure-dilatation as the mechanism driving oscillatory energy transfers across scales. Their findings indicated that the oscillation period diminishes with increasing wavenumber, underscoring the role of acoustic modes in mediating scale-dependent energy exchange. Praturi and Girimaji [28] noted that at high wavenumbers, compressive velocity fields carry more energy compared with their rotational counterparts, which is attributed to the lack of pressure action enforcing a divergence-free condition, thereby facilitating shock formation. Jagannathan and Donzis [25] and John and Donzis [29] investigated the scaling of global quantities with Reynolds and Mach numbers using numerical simulations and reported strong evidence supporting universality across different flow regimes. As an aside, for a nearly incompressible flow $[(\delta\rho)/\rho \to 0]$, Zank and Matthaeus [30] showed that both velocity and density fields follow $k^{-5/3}$ spectrum (also see Ref. [31]).

Several studies have examined interscale kinetic energy transfers in compressible turbulence using both coarse-graining and spectral techniques [22,23,28,32–36]. Aluie [37] argued that in the inertial range, the kinetic energy transfer is conservative and is dominated by local interactions. Aluie [38] derived scale-by-scale energy transfers in compressible turbulence. Similar relations have also been validated through simulations [23,33–35]. Wang *et al.* [33] showed that the solenoidal energy flux is nearly independent of the turbulent Mach number M_t, whereas the compressible flux grows with M_t because of the energy transfers from solenoidal to compressive modes. Luo *et al.* [34] decomposed subgrid-scale flux into solenoidal, compressible, and mixed-mode contributions. Though structure-function-based approaches have also been explored [39–41], a complete picture of kinetic-energy exchange between the solenoidal and compressive modes remains incomplete.

Graham *et al.* [35] developed a spectral framework to study energy transfers in compressible magnetohydrodynamics turbulence. They derived the transfer terms and separated them into advective and compressive contributions. Applying this approach to simulations for solar surface convection, they identified vortex stretching against magnetic tension as the primary mechanism driving small-scale dynamo action. Grete *et al.* [42] and Schmidt and Grete [23] used the density-weighted velocity $\mathbf{w} = \sqrt{\rho}\,\mathbf{u}$ (ρ is the density and $\mathbf{u}$ is the velocity) to analyze the kinetic-energy transfers using implicit large-eddy simulations. They also decomposed the cascade into advective and compressive components and showed that the velocity spectra, scaling exponents, and energy fluxes depend strongly on the nature of the external forcing. Their results support a transition from Kolmogorov to Burgers scaling for velocity spectra depending on M_t and on forcing composition.

As described earlier, the energy spectra of the rotational and compressive velocity components consistently show $k^{-5/3}$ and k^{-2} spectra, respectively. However, the multiscale energy transfers and fluxes for the compressible turbulence have not been computed in *detail*. Building on earlier works of Dar *et al.* [11], Schmidt and Grete [23], Graham *et al.* [35], and Verma [12], we derive formulas for the mode-to-mode transfers and the energy fluxes for compressible turbulence, including the rotational and compressive velocity components and internal energy. The mode-to-mode energy formalism presented in this work generalizes the formalism used for incompressible turbulence [11,12]. In addition, several properties of the energy fluxes for compressible turbulence resemble those for magnetohydrodynamic turbulence [12,31].

This paper presents the theoretical framework of mode-to-mode energy transfer for compressible turbulence, including the derivation of exact expressions for spectral energy fluxes. The structure of the paper is as follows: Section II presents the equations of compressible flows in spectral space. In Sec. III, we derive the mode-to-mode energy transfers for compressible flows. We define the corresponding kinetic-energy fluxes in Sec. IV. Section V describes the alternative formalism of mode-to-mode energy transfers and energy fluxes in terms of density-weighted velocity $\mathbf{w} = \sqrt{\rho}\mathbf{u}$. In Sec. VI, we compare our results with past ones. Finally, we conclude in Sec. VII.

II. COMPRESSIBLE EQUATIONS IN SPECTRAL SPACE

The equations for a compressible flow in nondimensional and tensorial form are [20]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij}\sigma - \tau_{ij}) = \rho F_i, \tag{2}$$

$$\frac{\partial E_T}{\partial t} + \frac{\partial}{\partial x_i}\left(u_i(E_T + \sigma) - \frac{1}{M_0^2 \mathrm{Pr}\mathrm{Re}_0(\gamma - 1)}\frac{\partial T}{\partial x_i} - u_j \tau_{ij}\right) = \rho u_i F_i, \tag{3}$$

where ρ, $\mathbf{u}$, σ, T, $\mathbf{F}$ are the density, velocity, pressure, temperature, and external force fields, respectively;

$$\tau_{ij} = \frac{1}{\mathrm{Re}_0}\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\partial_m u_m \delta_{ij}\right) \tag{4}$$

is the viscous stress tensor, and E_T is the total energy density, which is a sum of kinetic-energy (KE) density

$$E_u = \frac{\rho u^2}{2}, \tag{5}$$

and internal energy (IE) density

$$I = \frac{\sigma}{(\gamma - 1)}, \tag{6}$$

with $\gamma = C_p/C_v$ as the ratio of specific heat capacities at constant pressure and volume. We assume the fluid to be an ideal gas with the equation of state

$$\sigma = \frac{\rho T}{\gamma M_0^2}, \tag{7}$$

where M_0 is the reference Mach number. Equations (1)–(3) have been nondimensionalized using reference density ρ_0, temperature T_0, velocity u_0, and length l_0. The dimensionless parameters of the system are

$$\text{Reynolds number } \mathrm{Re}_0 = \frac{\rho_0 u_0 l_0}{\mu}, \tag{8}$$

$$\text{Reynolds number (Taylor microscale) } \mathrm{Re}_\lambda = \left(\frac{5}{3\mu\epsilon}\right)^{1/2} \rho_0 U^2, \tag{9}$$

$$\text{Mach number } M_0 = \frac{u_0}{C_s} = \frac{u_0}{\sqrt{\gamma R^* T_0}}, \tag{10}$$

$$\text{Turbulent Mach number } M_t = \frac{U}{C_s}, \tag{11}$$

$$\text{Prandtl number Pr} = \frac{\mu C_p}{K_c}, \tag{12}$$

where C_s is the sound speed; R^* is the gas constant; μ is the dynamic viscosity; ϵ is the mean viscous dissipation rate; U is the root-mean-square velocity; and K_c is the thermal conductivity [20,25]. When $\mathbf{F} = 0$, an integration of Eq. (3) over real space yields

$$\frac{d}{dt}\int d\mathbf{r} E_T(\mathbf{r}, t) = 0. \tag{13}$$

This result implies that the total energy E_T is conserved in the absence of an external force. Note, however, that the KE is transferred to the IE. In this paper, we analyze the scale-by-scale transfers of KE and IE.

Fourier decomposition helps in the scale-by-scale analysis of energy contents and transfers in the flow [31]. In the following discussion, we derive the KE transfers among the velocity Fourier modes. To make the KE quadratic, we rewrite it as $\mathbf{u}\cdot\mathbf{v}/2$ with

$$\mathbf{v} = \rho\mathbf{u}. \tag{14}$$

Hence, the modal energy is [35]

$$E_u(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[\mathbf{v}(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})]. \tag{15}$$

Here, $\mathbf{u}(\mathbf{k})$ and $\mathbf{v}(\mathbf{k})$ are the Fourier amplitudes of Fourier modes for $\mathbf{u}(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$ fields, respectively. A Fourier transform of Eqs. (1) and (2) yields (see Appendix A for details)

$$\frac{d}{dt}\mathbf{v}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{v}(\mathbf{p}) - i\mathbf{k}\sigma(\mathbf{k}) - \mathbf{d}(\mathbf{k}) + \mathbf{F}'(\mathbf{k}), \tag{16}$$

$$\frac{d}{dt}\mathbf{u}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{u}(\mathbf{p}) - \tilde{\boldsymbol{\sigma}}(\mathbf{k}) - \tilde{\mathbf{d}}(\mathbf{k}) + \mathbf{F}(\mathbf{k}), \tag{17}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$, and

$$\tilde{\boldsymbol{\sigma}} = \nabla\sigma/\rho, \quad \mathbf{d} = -\partial_j\tau_{ij}, \quad \tilde{\mathbf{d}} = \mathbf{d}/\rho, \quad \mathbf{F}' = \rho\mathbf{F}. \tag{18}$$

Using the above, we derive the following dynamical equation for the modal KE, $E_u(\mathbf{k})$,

$$\frac{d}{dt}E_u(\mathbf{k}) = T_u(\mathbf{k}) - Q_{I,u}(\mathbf{k}) - D_{I,u}(\mathbf{k}) + \mathcal{F}_u(\mathbf{k}), \tag{19}$$

where

$$T_u(\mathbf{k}) = \tfrac{1}{2}\sum_{\mathbf{p}}\mathrm{Im}[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\} + \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}^*(\mathbf{k})\}], \tag{20}$$

$$Q_{I,u}(\mathbf{k}) = -\tfrac{1}{2}\mathrm{Im}[\sigma(\mathbf{k})\{\mathbf{k}\cdot\mathbf{u}^*(\mathbf{k})\}] + \tfrac{1}{2}\mathrm{Re}\{\tilde{\boldsymbol{\sigma}}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})\}, \tag{21}$$

$$D_{I,u}(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[\mathbf{d}(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k}) + \tilde{\mathbf{d}}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})], \tag{22}$$

$$\mathcal{F}_u(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[\mathbf{F}'(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k}) + \mathbf{F}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})]. \tag{23}$$

The interpretation of Eqs. (20)–(23) is as follows:

(1) $\sum_{\mathbf{p}}\mathrm{Im}[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\}]$: KE transfer from $\mathbf{v}(\mathbf{p})$ to $\mathbf{u}(\mathbf{k})$.

(2) $\sum_{\mathbf{p}}\mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}^*(\mathbf{k})\}]$: KE transfer from $\mathbf{u}(\mathbf{p})$ to $\mathbf{v}(\mathbf{k})$.

(3) $Q_{I,u}(\mathbf{k})$: At wavenumber $\mathbf{k}$, transfer of KE to IE via pressure-dilatation or the work done by pressure [43].

(4) $D_{I,u}(\mathbf{k})$: Viscous dissipation of KE at wavenumber $\mathbf{k}$. This lost KE reaches IE.
(5) $\mathcal{F}_u(\mathbf{k})$: At wavenumber $\mathbf{k}$, KE injection rate by the external force $\mathbf{F}$.

The unit interactions of items 1 and 2 involve wavenumbers of a triad, but those in items 3, 4, and 5 include wavenumbers beyond those in the triad because $\sigma(\mathbf{k})$, $\mathbf{d}(\mathbf{k})$, $\tilde{\sigma}(\mathbf{k})$, $\tilde{\mathbf{d}}(\mathbf{k})$, and $\mathbf{F}'(\mathbf{k})$ involve convolutions.

We focus on a single triad without $\mathbf{F}$ to obtain detailed scale-by-scale energy transfers. Following Kraichnan [10], Dar *et al.* [11], and Verma [12], we consider a triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ with $\mathbf{k}' + \mathbf{p} + \mathbf{q} = 0$ [11,31]. Note that $\mathbf{k}' = -\mathbf{k}$. For this triad,

$$\frac{d}{dt}E_u(\mathbf{k}') = S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) - Q_{I,u}(\mathbf{k}') - D_{I,u}(\mathbf{k}'), \tag{24}$$

where

$$S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}(\mathbf{k}')\} - \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}(\mathbf{k}')\}] \\ - \tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{v}(\mathbf{q})\cdot\mathbf{u}(\mathbf{k}')\} - \{\mathbf{q}\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{u}(\mathbf{q})\cdot\mathbf{v}(\mathbf{k}')\}] \tag{25}$$

is the combined KE transfer to wavenumber $\mathbf{k}'$ from wavenumbers $\mathbf{p}$ and $\mathbf{q}$. We can show that (see Appendix A for a proof)

$$S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) + S^{uu}(\mathbf{p}|\mathbf{k}',\mathbf{q}) + S^{uu}(\mathbf{q}|\mathbf{p},\mathbf{k}') = 0. \tag{26}$$

Therefore,

$$\frac{d}{dt}[E_u(\mathbf{k}') + E_u(\mathbf{p}) + E_u(\mathbf{q})] = -[Q_{I,u}(\mathbf{k}') + Q_{I,u}(\mathbf{p}) + Q_{I,u}(\mathbf{q}) \\ + D_{I,u}(\mathbf{k}') + D_{I,u}(\mathbf{p}) + D_{I,u}(\mathbf{q})] \neq 0. \tag{27}$$

The right-hand side of Eq. (27) does not contain $S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})$, implying that the net triadic KE is conserved for the triadic interactions. This is the detailed conservation law for compressible hydrodynamics, which is a generalization of a similar law for incompressible hydrodynamics [10]. Note, however, that the net KE of the triad is not conserved as $Q_{I,u}(\mathbf{k}')$ and $D_{I,u}(\mathbf{k}')$ transfer KE to IE.

In the next section, we formulate mode-to-mode energy transfers in compressible turbulence.

III. MODE-TO-MODE ENERGY TRANSFERS IN COMPRESSIBLE TURBULENCE

As shown in Sec. II, for a triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$, $S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})$ of Eq. (25) provides the net KE transfer from wavenumbers $\mathbf{p}$ and $\mathbf{q}$ to wavenumber $\mathbf{k}'$. In this section, we derive individual KE transfer from wavenumber $\mathbf{p}$ to wavenumber $\mathbf{k}'$ and from wavenumber $\mathbf{q}$ to wavenumber $\mathbf{k}'$. In this section, we prove that for a triad $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ with $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, the mode-to-mode kinetic-energy transfer from wavenumber $\mathbf{b}$ to wavenumber $\mathbf{a}$ with the mediation of wavenumber $\mathbf{c}$ is given by

$$S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c}) = -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{a}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{v}(\mathbf{b})\cdot\mathbf{u}(\mathbf{a})\} - \{\mathbf{b}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{u}(\mathbf{b})\cdot\mathbf{v}(\mathbf{a})\}]. \tag{28}$$

This formula is a generalization of a similar framework for incompressible turbulence derived by Dar *et al.* [11] and Verma [12].

We prove the above statement [Eq. (28)] using a similar approach to that used for incompressible turbulence. By definition, the mode-to-mode KE transfer formula must satisfy the following properties:

(1) The combined energy transfer is a sum of individual mode-to-mode energy transfers, i.e.,

$$S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c}) + S^{uu}(\mathbf{a}|\mathbf{c}|\mathbf{b}) = S^{uu}(\mathbf{a}|\mathbf{b},\mathbf{c}). \tag{29}$$

The above property is a logical consequence of its definition [Eq. (25)]. Similarly, we derive two more equations for $S^{uu}(\mathbf{b}|\mathbf{a},\mathbf{c})$ and $S^{uu}(\mathbf{c}|\mathbf{a},\mathbf{b})$.

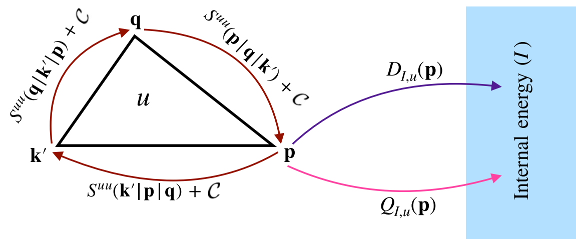

FIG. 1. Schematic illustration of mode-to-mode energy transfers $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ in a triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$. Here, $Q_{I,u}(\mathbf{p})$ and $D_{I,u}(\mathbf{p})$ represent pressure dilatation and viscous dissipation, respectively. $\mathcal{C}$ is the arbitrary circulatory transfer that can be added to $S^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$.

(2) The mode-to-mode transfer from wavenumber **a** to **b** is equal and opposite to that from **b** to **a**, i.e.,

$$S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c}) = -S^{uu}(\mathbf{b}|\mathbf{a}|\mathbf{c}). \tag{30}$$

We prove the above statement as follows:

$$\begin{aligned} S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c}) + S^{uu}(\mathbf{b}|\mathbf{a}|\mathbf{c}) = -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{a}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{v}(\mathbf{b})\cdot\mathbf{u}(\mathbf{a})\} - \{\mathbf{b}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{u}(\mathbf{b})\cdot\mathbf{v}(\mathbf{a})\} \\ +\mathbf{b}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{v}(\mathbf{a})\cdot\mathbf{u}(\mathbf{b})\} - \{\mathbf{a}\cdot\mathbf{u}(\mathbf{c})\}\{\mathbf{u}(\mathbf{a})\cdot\mathbf{v}(\mathbf{b})\}] = 0. \end{aligned} \tag{31}$$

There are two additional equations for $S^{uu}(\mathbf{b}|\mathbf{c}|\mathbf{a})$ and $S^{uu}(\mathbf{c}|\mathbf{a}|\mathbf{b})$.

Thus, we have six equations for six unknowns $S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c})$, $S^{uu}(\mathbf{a}|\mathbf{c}|\mathbf{b})$, $S^{uu}(\mathbf{b}|\mathbf{a}|\mathbf{c})$, $S^{uu}(\mathbf{b}|\mathbf{c}|\mathbf{a})$, $S^{uu}(\mathbf{c}|\mathbf{a}|\mathbf{b})$, $S^{uu}(\mathbf{c}|\mathbf{b}|\mathbf{a})$. It is easy to verify that $S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c})$ of the form given in Eq. (28) satisfies Eqs. (29) and (30). Therefore, $S^{uu}(\mathbf{a}|\mathbf{b}|\mathbf{c})$ is indeed the mode-to-mode transfer from wavenumber **b** to wavenumber **a** with the mediation of wavenumber **c**. Unfortunately, the above solution is not unique. Following Dar *et al.* [11], we can add a circulatory transfer $\mathcal{C}$ to the formula (see Fig. 1). Fortunately, these circulatory transfers do not affect the energy flux, and hence, we can set $\mathcal{C} = 0$, just like a gauge field in electromagnetism [11]. Figure 1 illustrates the mode-to-mode energy transfers for a triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$. It also exhibits the loss of modal KE, $E_u(\mathbf{p})$, to IE via pressure dilatation $Q_{I,u}(\mathbf{p})$ and viscous dissipation $D_{I,u}(\mathbf{p})$.

We gain further insights into the system by dividing the velocity field into its rotational (or solenoidal) component $\mathbf{u}_R$ and compressive component $\mathbf{u}_C$ [20]. In Fourier space,

$$\mathbf{u}(\mathbf{k}) = \mathbf{u}_R(\mathbf{k}) + \mathbf{u}_C(\mathbf{k}). \tag{32}$$

Note that $\mathbf{u}_C(\mathbf{k})$ is along $\mathbf{k}$ and $\mathbf{u}_R(\mathbf{k})$ is perpendicular to $\mathbf{k}$, as illustrated in Fig. 2. The modal kinetic energies for these components are

$$E_\alpha(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[\mathbf{v}_\alpha(\mathbf{k})\cdot\mathbf{u}_\alpha^*(\mathbf{k})], \tag{33}$$

where $\alpha = R, C$, whereas the total kinetic energy is

$$E_u(\mathbf{k}) = E_R(\mathbf{k}) + E_C(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[\mathbf{v}_R(\mathbf{k})\cdot\mathbf{u}_R^*(\mathbf{k})] + \tfrac{1}{2}\mathrm{Re}[\mathbf{v}_C(\mathbf{k})\cdot\mathbf{u}_C^*(\mathbf{k})]. \tag{34}$$

The dynamical equation for $E_\alpha(\mathbf{k}')$ is

$$\begin{aligned} \frac{d}{dt}E_\alpha(\mathbf{k}') &= T_{\alpha\alpha} + T_{\alpha\beta} - Q_{I,\alpha}(\mathbf{k}') - D_{I,\alpha}(\mathbf{k}') + \mathcal{F}_\alpha(\mathbf{k}') \\ &= \sum_{\mathbf{p}} S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \sum_{\mathbf{p}} S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) - Q_{I,\alpha}(\mathbf{k}') - D_{I,\alpha}(\mathbf{k}') + \mathcal{F}_\alpha(\mathbf{k}'), \end{aligned} \tag{35}$$

where

$$S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_\alpha(\mathbf{p})\cdot\mathbf{u}_\alpha(\mathbf{k}')\} - \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}_\alpha(\mathbf{p})\cdot\mathbf{v}_\alpha(\mathbf{k}')\}], \tag{36}$$

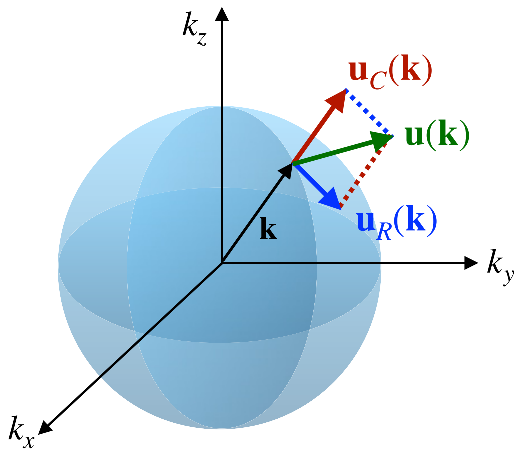

FIG. 2. Decomposition of a velocity mode $\mathbf{u}(\mathbf{k})$ into its rotational component $\mathbf{u}_R(\mathbf{k})$ and compressive component $\mathbf{u}_C(\mathbf{k})$.

$$S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_\beta(\mathbf{p})\cdot\mathbf{u}_\alpha(\mathbf{k}')\} - \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}_\beta(\mathbf{p})\cdot\mathbf{v}_\alpha(\mathbf{k}')\}], \tag{37}$$

$$Q_{I,R}(\mathbf{k}') = \tfrac{1}{2}\mathrm{Re}[\tilde{\boldsymbol{\sigma}}(\mathbf{k}')\cdot\mathbf{v}_R^*(\mathbf{k}')], \tag{38}$$

$$Q_{I,C}(\mathbf{k}') = \tfrac{1}{2}\mathrm{Re}[\tilde{\boldsymbol{\sigma}}(\mathbf{k}')\cdot\mathbf{v}_C^*(\mathbf{k}')] - \tfrac{1}{2}\mathrm{Im}[\sigma(\mathbf{k}')\{\mathbf{k}'\cdot\mathbf{u}_C^*(\mathbf{k}')\}], \tag{39}$$

$$D_{I,\alpha}(\mathbf{k}') = \tfrac{1}{2}\mathrm{Re}[\mathbf{d}_\alpha(\mathbf{k}')\cdot\mathbf{u}_\alpha^*(\mathbf{k}') + \tilde{\mathbf{d}}_\alpha(\mathbf{k}')\cdot\mathbf{v}_\alpha^*(\mathbf{k}')], \tag{40}$$

$$\mathcal{F}_\alpha(\mathbf{k}') = \tfrac{1}{2}\mathrm{Re}[\mathbf{F}'_\alpha(\mathbf{k}')\cdot\mathbf{u}_\alpha^*(\mathbf{k}') + \mathbf{F}_\alpha(\mathbf{k}')\cdot\mathbf{v}_\alpha^*(\mathbf{k}')]. \tag{41}$$

Note that

$$S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\alpha\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0, \tag{42}$$

$$S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\beta\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0. \tag{43}$$

Interestingly, $S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ and its variants satisfy Eqs. (29) and (30). Therefore, following the same arguments as before, we can deduce that for the component α (R or C), $S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents the energy-transfer rate from wavenumber $\mathbf{p}$ to wavenumber $\mathbf{k}$' with the mediation of wavenumber $\mathbf{q}$. Here, the giver and receiver modes belong to $\mathbf{u}_\alpha$ field, but the mediator is the full $\mathbf{u}$ field. We illustrate these energy transfers in Fig. 3. Using Eq. (42), we derive that

$$S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\alpha\alpha}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{\alpha\alpha}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + S^{\alpha\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + S^{\alpha\alpha}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + S^{\alpha\alpha}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = 0, \tag{44}$$

which implies that $E_\alpha(\mathbf{k}') + E_\alpha(\mathbf{p}) + E_\alpha(\mathbf{q})$ is *partially* conserved for the energy transfers along this channel.

In addition, $S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ and its variants satisfy Eqs. (29) and (43). Therefore, using similar reasoning, we interpret $S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ as the energy-transfer rate from $\mathbf{u}_\beta(\mathbf{p})$ to $\mathbf{u}_\alpha(\mathbf{k}')$ with the mediation of $\mathbf{u}(\mathbf{q})$. Here, the giver and receiver modes belong to $\mathbf{u}_\beta$ and $\mathbf{u}_\alpha$ fields, respectively, but the mediator is the full $\mathbf{u}$ field. Here, $\alpha \neq \beta$. These cross transfers are illustrated in Fig. 3. Note that an application of Eq. (43) yields

$$\begin{aligned} &S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\alpha\beta}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{\alpha\beta}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + S^{\alpha\beta}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + S^{\alpha\beta}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + S^{\alpha\beta}(\mathbf{q}|\mathbf{p}|\mathbf{k}') \\ &\quad + S^{\beta\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + S^{\beta\alpha}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) + S^{\beta\alpha}(\mathbf{p}|\mathbf{q}|\mathbf{k}') + S^{\beta\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) + S^{\beta\alpha}(\mathbf{q}|\mathbf{k}'|\mathbf{p}) + S^{\beta\alpha}(\mathbf{q}|\mathbf{p}|\mathbf{k}') = 0, \end{aligned} \tag{45}$$

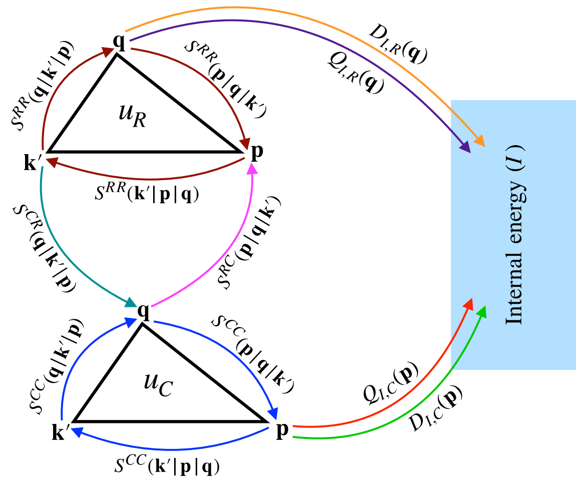

FIG. 3. Mode-to-mode energy-transfer terms between decomposed kinetic energies with rotational and compressive velocity components, $\mathbf{u}_R$ and $\mathbf{u}_C$, respectively: $S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents the energy-transfer rate from mode $\mathbf{u}_\alpha(\mathbf{p})$ to mode $\mathbf{u}_\alpha(\mathbf{k}')$ with the mediation of full $\mathbf{u}(\mathbf{q})$, while $S^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents the energy transfer rate from mode $\mathbf{u}_\beta(\mathbf{p})$ to mode $\mathbf{u}_\alpha(\mathbf{k}')$ with the mediation of full $\mathbf{u}(\mathbf{q})$. $Q_{I,\alpha}(\mathbf{p})$ and $D_{I,\alpha}(\mathbf{p})$ denote pressure dilatation and viscous dissipation of KE at wavenumber $\mathbf{p}$ to internal energy (I). Here, $\alpha, \beta = R, C$ where R and C represent the rotational and compressive kinetic-energy modes.

which implies a conservation of KE by these interactions. The above transfers redistribute the KE among the triadic modes without any gain or loss. We can also write the mode-to-mode transfer terms in terms of $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ with $\mathbf{k} = \mathbf{p} + \mathbf{q}$ as

$$S^{\alpha\alpha}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_\alpha(\mathbf{p})\cdot\mathbf{u}_\alpha^*(\mathbf{k})\} + \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}_\alpha(\mathbf{p})\cdot\mathbf{v}_\alpha^*(\mathbf{k})\}], \tag{46}$$

$$S^{\alpha\beta}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_\beta(\mathbf{p})\cdot\mathbf{u}_\alpha^*(\mathbf{k})\} + \{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}_\beta(\mathbf{p})\cdot\mathbf{v}_\alpha^*(\mathbf{k})\}]. \tag{47}$$

At this point, it is important to compare the above formulas with those for incompressible flows, where $\mathbf{u}_C = 0$ and $\rho = \text{const}$. Since $\mathbf{u}_C = 0$, we easily deduce that

$$S^{CC}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = 0; \quad Q_{I,C}(\mathbf{k}) = 0, \tag{48}$$

and

$$\begin{aligned} S^{RR}(\mathbf{k}|\mathbf{p}|\mathbf{q}) &= \frac{\rho}{2}\mathrm{Im}[\{\mathbf{k}\cdot\mathbf{u}_R(\mathbf{q})\}\{\mathbf{u}_R(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k})\} + \{(\mathbf{k}-\mathbf{q})\cdot\mathbf{u}_R(\mathbf{q})\}\{\mathbf{u}_R(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k})\}], \\ &= \rho\mathrm{Im}\big[\{\mathbf{k}\cdot\mathbf{u}_R(\mathbf{q})\}\{\mathbf{u}_R(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k})\}, \end{aligned} \tag{49}$$

which is same as that derived by Dar *et al.* [11]. Similarly, we show that

$$Q_{I,R}(\mathbf{k}) = \tfrac{1}{2}\mathrm{Re}[i\sigma(\mathbf{k})\{\mathbf{k}\cdot\mathbf{u}^*(\mathbf{k})\}] = 0 \tag{50}$$

for an incompressible flow. Note that the internal energy, I, is not considered in an incompressible flow. It is implicitly assumed that the viscous dissipation increases the internal energy, but this conversion is not accounted for. Thus, the energy transfers in compressible turbulence reduce to those in incompressible turbulence where $\mathbf{u}_C = 0$ and ρ is constant.

In the following section, Sec. IV, we describe the energy fluxes in compressible turbulence.

IV. ENERGY FLUXES IN COMPRESSIBLE TURBULENCE

The mode-to-mode energy transfers described in earlier sections provide valuable insights into the turbulence dynamics. However, these transfers exhibit substantial fluctuations, obscuring underlying patterns. Therefore, researchers employ energy flux, shell-to-shell transfers, and ring-to-ring transfers, which involve sums of many unit energy transfers [$S^{\alpha\alpha}(\mathbf{k}|\mathbf{p}|\mathbf{q})$ and $S^{\alpha\beta}(\mathbf{k}|\mathbf{p}|\mathbf{q})$] [31]. In this section, we derive the energy fluxes in compressible turbulence.

Using the mode-to-mode energy transfers, we define energy transfers from $\mathbf{u}_\beta$ in the wavenumber region Y to $\mathbf{u}_\alpha$ in the wavenumber region X as [12,31]

$$\mathcal{T}_{\alpha,X}^{\beta,Y} = \sum_{\mathbf{k}\in X}\sum_{\mathbf{p}\in Y} S^{\alpha\beta}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \tag{51}$$

In $\mathcal{T}_{\alpha,X}^{\beta,Y}$, the superscript refers to the giver field, whereas the subscript refers to the receiver field. In particular, the shell-to-shell transfer of $\mathbf{u}_\beta$ from the shell m to $\mathbf{u}_\alpha$ of the shell n is [44]

$$\mathcal{T}_{\alpha,n}^{\beta,m} = \sum_{\mathbf{k}\in s_n}\sum_{\mathbf{p}\in s_m} S^{\alpha\beta}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \tag{52}$$

The shell-to-shell energy transfers in incompressible hydrodynamic turbulence is local in the inertial range. That is, the maximal energy transfer is from shell m to shell $m+1$ of the inertial range [13–18]. We plan to test locality in compressible turbulence in the future. Appendix B contains preliminary results on the shell-to-shell energy transfers.

Using this broad energy-transfer framework, we further define the energy flux for the rotational and compressive components as follows: The energy flux $\Pi_R(K)$ [$\Pi_C(K)$] is the net energy transfer from the $\mathbf{u}_R$ ($\mathbf{u}_C$) modes inside the wavenumber sphere to the $\mathbf{u}_R$ ($\mathbf{u}_C$) modes outside the sphere of radius K, i.e.,

$$\Pi_R(K) = \Pi_{R>}^{R<}(K) = \sum_{k>K}\sum_{p\leqslant K} S^{RR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{53}$$

$$\Pi_C(K) = \Pi_{C>}^{C<}(K) = \sum_{k>K}\sum_{p\leqslant K} S^{CC}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \tag{54}$$

Here, "<" denotes modes within the wave-number sphere, and ">" denotes modes outside it. In addition to the above, we define fluxes for the cross transfers between $\mathbf{u}_R$ and $\mathbf{u}_C$:

$$\Pi_{C<}^{R<}(K) = \sum_{k\leqslant K}\sum_{p\leqslant K} S^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{55}$$

$$\Pi_{C>}^{R>}(K) = \sum_{k>K}\sum_{p>K} S^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{56}$$

$$\Pi_{C>}^{R<}(K) = \sum_{k>K}\sum_{p\leqslant K} S^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{57}$$

$$\Pi_{R>}^{C<}(K) = \sum_{k>K}\sum_{p\leqslant K} S^{RC}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \tag{58}$$

For example, $\Pi_{C>}^{R<}(K)$ is the energy transfers from $\mathbf{u}_R$ modes inside the sphere of radius K to $\mathbf{u}_C$ modes outside the sphere, whereas $\Pi_{C<}^{R<}(K)$ is the energy transfers from $\mathbf{u}_R$ modes to $\mathbf{u}_C$ modes inside the sphere. These fluxes are illustrated in Fig. 4. We also define the energy fluxes using the nonlinear energy transfers $T_{\alpha\alpha}(\mathbf{k})$ and $T_{\alpha\beta}(\mathbf{k})$ as follows [7,31,45]:

$$\Pi_\alpha(K) = -\sum_{k\leqslant K} T_{\alpha\alpha}(\mathbf{k}), \tag{59}$$

$$\Pi_\beta^{\alpha<}(K) = \Pi_{\beta<}^{\alpha<}(K) + \Pi_{\beta>}^{\alpha<}(K) = -\sum_{k\leqslant K} T_{\alpha\beta}(\mathbf{k}). \tag{60}$$

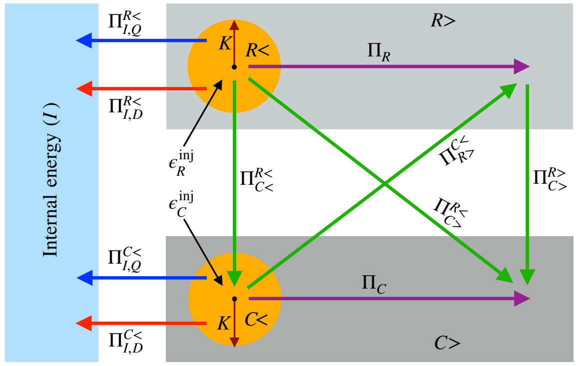

FIG. 4. Various fluxes in compressible turbulence: Rotational and compressive energy fluxes (Π_R, Π_C), cross fluxes ($\Pi_{C<}^{R<}$, $\Pi_{C>}^{R<}$, $\Pi_{C>}^{R>}$, $\Pi_{R>}^{C<}$), pressure dilatation ($\Pi_{I,Q}^{R}$, $\Pi_{I,Q}^{C}$), viscous dissipation ($\Pi_{I,D}^{R}$, $\Pi_{I,D}^{C}$), and energy-injection rates ($\epsilon_R^{\rm inj}$, $\epsilon_C^{\rm inj}$).

Equation (60) represents the net energy transfers from $\mathbf{u}_\alpha$ modes inside the sphere to all the $\mathbf{u}_\beta$ modes. Note that $\Pi_\alpha(K=\infty)=0$ because

$$\Pi_\alpha(K=\infty)=\sum_{0<k'<\infty}\sum_{0<p<\infty}S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})=\tfrac{1}{2}\sum_{0<k'<\infty}\sum_{0<p<\infty}[S^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})+S^{\alpha\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q})]=0. \tag{61}$$

We use Eqs. (59) and (60) to calculate energy fluxes efficiently. A fraction of kinetic energy is transferred to the internal energy via pressure dilatation Q and viscous dissipation D, for which we define the energy fluxes $\Pi_{I,Q}^{\alpha<}(K)$ and $\Pi_{I,D}^{\alpha<}(K)$ as follows:

$$\Pi_{I,Q}^{\alpha<}(K)=\sum_{k\leqslant K}Q_{I,\alpha}(\mathbf{k}), \tag{62}$$

$$\Pi_{I,D}^{\alpha<}(K)=\sum_{k\leqslant K}D_{I,\alpha}(\mathbf{k}). \tag{63}$$

We illustrate these transfers in Fig. 4. Using the aforementioned formulas, we write the following equations:

$$\frac{d}{dt}\sum_{k\leqslant K}E_R(\mathbf{k})=-\Pi_R(K)-\Pi_C^{R<}(K)-\Pi_{I,Q}^{R<}(K)-\Pi_{I,D}^{R<}(K)+\epsilon_R^{\rm inj}, \tag{64}$$

$$\frac{d}{dt}\sum_{k\leqslant K}E_C(\mathbf{k})=-\Pi_C(K)-\Pi_R^{C<}(K)-\Pi_{I,Q}^{C<}(K)-\Pi_{I,D}^{C<}(K)+\epsilon_C^{\rm inj}, \tag{65}$$

with

$$\sum_{k\leqslant k_f}\mathcal{F}_R(k)=\epsilon_R^{\rm inj}, \tag{66}$$

$$\sum_{k\leqslant k_f}\mathcal{F}_C(k)=\epsilon_C^{\rm inj}, \tag{67}$$

where $\epsilon_R^{\rm inj}$ and $\epsilon_C^{\rm inj}$ are the energy injection rates to the rotational and compressive components, respectively; and k_f is the forcing wave-number band. For a steady state, $d/dt\sum_{k\leqslant K}E_\alpha(\mathbf{k})=0$, we obtain the following exact relations for $K>k_f$:

$$\Pi_R(K)+\Pi_C^{R<}(K)+\Pi_{I,Q}^{R<}(K)+\Pi_{I,D}^{R<}(K)=\epsilon_R^{\rm inj}=I_1=\text{const}, \tag{68}$$

$$\Pi_C(K)+\Pi_R^{C<}(K)+\Pi_{I,Q}^{C<}(K)+\Pi_{I,D}^{C<}(K)=\epsilon_C^{\rm inj}=I_2=\text{const}. \tag{69}$$

These exact relations, which are similar to those for MHD turbulence and similar systems [31,45,46], hold for K's in the inertial-dissipation range (beyond the forcing wavenumbers).

Equation (51) and the energy flux formulas are similar to those used for incompressible flows, such as magnetohydrodynamic turbulence [11,31]. This similarity is because of the accurate identification of the giver and receiver modes that capture the common features of energy transfers irrespective of details. The formulas for the mode-to-mode energy transfers for the compressible and incompressible turbulence are different in form, but they are very similar in spirit. Hence, the above energy-transfer framework provides valuable universal tools for turbulence analysis.

V. MODE-TO-MODE ENERGY TRANSFERS AND ENERGY FLUXES FOR $\mathbf{w} = \sqrt{\rho}\mathbf{u}$

The density-weighted velocity $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ provides an alternative to define quadratic kinetic energy in compressible turbulence [19,20,23,27,28,47–50]. In particular, Kida and Orszag [20,21], Miura and Kida [27], Schmidt and Grete [23], Praturi and Girimaji [28], and Grete *et al.* [42] used $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ to derive energy spectra and quantify interscale energy transfers in compressible turbulence. In this section, we derive the mode-to-mode energy-transfer formalism based on $\mathbf{w}$. The early part of the derivation is similar to Miura and Kida [27], Schmidt and Grete [23], and Praturi and Girimaji [28]. In terms of $\mathbf{w}$, the kinetic-energy density in quadratic form is [20,21,23,27]

$$\overline{E}_u = \tfrac{1}{2}|\mathbf{w}|^2. \tag{70}$$

The dynamical equation for $\mathbf{w}$ is

$$\frac{d}{dt}\mathbf{w} = \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{w} = \sqrt{\rho}\frac{\partial \mathbf{u}}{\partial t} + \frac{\mathbf{u}}{2\sqrt{\rho}}\frac{\partial \rho}{\partial t} + \sqrt{\rho}(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\mathbf{u}}{2\sqrt{\rho}}(\mathbf{u}\cdot\nabla)\rho. \tag{71}$$

Using Eqs. (1) and (2), we can rewrite Eq. (71) as [20]

$$\frac{\partial \mathbf{w}}{\partial t} = -(\mathbf{u}\cdot\nabla)\mathbf{w} - \tfrac{1}{2}\mathbf{w}(\nabla\cdot\mathbf{u}) - \frac{1}{\sqrt{\rho}}\nabla\sigma + \frac{1}{\mathrm{Re}_0\sqrt{\rho}}\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right] + \sqrt{\rho}\mathbf{F}, \tag{72}$$

whose Fourier transformation is

$$\frac{d}{dt}\mathbf{w}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{w}(\mathbf{p}) - \frac{i}{2}\sum_{\mathbf{p}}\{\mathbf{q}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{w}(\mathbf{p}) - \overline{\boldsymbol{\sigma}}(\mathbf{k}) - \overline{\mathbf{d}}(\mathbf{k}) + \overline{\mathbf{F}}(\mathbf{k}), \tag{73}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$ and

$$\overline{\boldsymbol{\sigma}} = \frac{\nabla\sigma}{\sqrt{\rho}}, \quad \overline{\mathbf{d}} = -\frac{1}{\mathrm{Re}_0\sqrt{\rho}}\left[\nabla^2\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\right], \quad \overline{\mathbf{F}} = \sqrt{\rho}\mathbf{F}. \tag{74}$$

Equation (73) can be rearranged as

$$\frac{d}{dt}\mathbf{w}(\mathbf{k}) = -\frac{i}{2}\sum_{\mathbf{p}}\{(\mathbf{k}+\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\mathbf{w}(\mathbf{p}) - \overline{\boldsymbol{\sigma}}(\mathbf{k}) - \overline{\mathbf{d}}(\mathbf{k}) + \overline{\mathbf{F}}(\mathbf{k}). \tag{75}$$

We take the dot product of Eq. (75) with $\mathbf{w}^*(\mathbf{k})$, which yields

$$\mathbf{w}^*(\mathbf{k})\cdot\frac{d}{dt}\mathbf{w}(\mathbf{k}) = -\frac{i}{2}\sum_{\mathbf{p}}\{(\mathbf{k}+\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p})\cdot\mathbf{w}^*(\mathbf{k})\} - [\overline{\boldsymbol{\sigma}}(\mathbf{k}) + \overline{\mathbf{d}}(\mathbf{k}) - \overline{\mathbf{F}}(\mathbf{k})]\cdot\mathbf{w}^*(\mathbf{k}). \tag{76}$$

After adding Eq. (76) with its complex conjugate, we obtain the dynamical equation for the modal kinetic energy $E_u(\mathbf{k}) = |\mathbf{w}(\mathbf{k})|^2/2$ as [23,27,28]

$$\frac{d}{dt}\overline{E}_u(\mathbf{k}) = \overline{T}_u(\mathbf{k}) - \overline{Q}_{I,u}(\mathbf{k}) - \overline{D}_{I,u}(\mathbf{k}) + \overline{\mathcal{F}}_u(\mathbf{k}). \tag{77}$$

Here,

$$\overline{T}_u(\mathbf{k}) = \tfrac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{(\mathbf{k}+\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p})\cdot\mathbf{w}^*(\mathbf{k})\}], \tag{78}$$

$$\overline{Q}_{I,u}(\mathbf{k}) = \mathrm{Re}[\overline{\sigma}(\mathbf{k})\cdot\mathbf{w}^*(\mathbf{k})], \tag{79}$$

$$\overline{D}_{I,u}(\mathbf{k}) = \mathrm{Re}[\overline{\mathbf{d}}(\mathbf{k})\cdot\mathbf{w}^*(\mathbf{k})], \tag{80}$$

$$\overline{\mathcal{F}}_u(\mathbf{k}) = \mathrm{Re}[\overline{\mathbf{F}}(\mathbf{k})\cdot\mathbf{w}^*(\mathbf{k})]. \tag{81}$$

For a single triad $(\mathbf{k}', \mathbf{p}, \mathbf{q})$ with $\mathbf{k}'+\mathbf{p}+\mathbf{q}=0$ ($\mathbf{k}'=-\mathbf{k}$) [11,31],

$$\frac{d}{dt}\overline{E}_u(\mathbf{k}') = \overline{S}^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) - \overline{Q}_{I,u}(\mathbf{k}') - \overline{D}_{I,u}(\mathbf{k}') + \overline{\mathcal{F}}_u(\mathbf{k}'), \tag{82}$$

where

$$\overline{S}^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}'-\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p})\cdot\mathbf{w}(\mathbf{k}')\}] - \tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}'-\mathbf{q})\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{w}(\mathbf{q})\cdot\mathbf{w}(\mathbf{k}')\}] \tag{83}$$

is the combined KE transfer to wavenumber $\mathbf{k}'$ from wavenumbers $\mathbf{p}$ and $\mathbf{q}$. Interestingly, $\overline{S}^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})$ follows the detailed conservation law similar to Eq. (26). Following the same steps as in Sec. III, we derive the mode-to-mode energy transfers for this framework. Here,

$$\overline{S}^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}'-\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p})\cdot\mathbf{w}(\mathbf{k}')\}] \tag{84}$$

is the mode-to-mode KE transfer from wavenumber $\mathbf{p}$ to wavenumber $\mathbf{k}'$ with the mediation of wavenumber $\mathbf{q}$. It can be shown that

$$\overline{S}^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \overline{S}^{uu}(\mathbf{k}'|\mathbf{q}|\mathbf{p}) = \overline{S}^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q}), \tag{85}$$

$$\overline{S}^{uu}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \overline{S}^{uu}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0, \tag{86}$$

which have the same form as Eqs. (29) and (30). Thus, Eq. (84) is equivalent to Eq. (28) derived in Sec. III using $\mathbf{v}=\rho\mathbf{u}$ and $\mathbf{u}$ variables.

As described in Sec. III, we divide the velocity field $\mathbf{w}$ into its rotational and compressive components. The modal kinetic energies for these components are [20,21,27]

$$\overline{E}_\alpha(\mathbf{k}) = \tfrac{1}{2}|\mathbf{w}_\alpha(\mathbf{k})|^2, \tag{87}$$

where $\alpha = R, C$, and the total kinetic energy is

$$\overline{E}_u(\mathbf{k}) = \overline{E}_R(\mathbf{k}) + \overline{E}_C(\mathbf{k}) = \tfrac{1}{2}|\mathbf{w}_R(\mathbf{k})|^2 + \tfrac{1}{2}|\mathbf{w}_C(\mathbf{k})|^2. \tag{88}$$

The dynamical equation for $\overline{E}_\alpha(\mathbf{k}')$ is

$$\begin{aligned}\frac{d}{dt}\overline{E}_\alpha(\mathbf{k}') &= \overline{T}_{\alpha\alpha} + \overline{T}_{\alpha\beta} - \overline{Q}_{I,\alpha}(\mathbf{k}') - \overline{D}_{I,\alpha}(\mathbf{k}') + \overline{\mathcal{F}}_\alpha(\mathbf{k}')\\ &= \sum_{\mathbf{p}}\overline{S}^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \sum_{\mathbf{p}}\overline{S}^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) - \overline{Q}_{I,\alpha}(\mathbf{k}') - \overline{D}_{I,\alpha}(\mathbf{k}') + \overline{\mathcal{F}}_\alpha(\mathbf{k}'),\end{aligned} \tag{89}$$

where

$$\overline{S}^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}'-\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}_\alpha(\mathbf{p})\cdot\mathbf{w}_\alpha(\mathbf{k}')\}, \tag{90}$$

$$\overline{S}^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) = -\tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}'-\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}_\beta(\mathbf{p})\cdot\mathbf{w}_\alpha(\mathbf{k}')\}, \tag{91}$$

$$\overline{Q}_{I,\alpha}(\mathbf{k}') = \mathrm{Re}[\overline{\boldsymbol{\sigma}}_\alpha(\mathbf{k}')\cdot\mathbf{w}^*_\alpha(\mathbf{k}')], \tag{92}$$

$$\overline{D}_{I,\alpha}(\mathbf{k}') = \mathrm{Re}[\overline{\mathbf{d}}_\alpha(\mathbf{k}')\cdot\mathbf{w}^*_\alpha(\mathbf{k}')], \tag{93}$$

$$\overline{\mathcal{F}}_\alpha(\mathbf{k}') = \mathrm{Re}[\overline{\mathbf{F}}_\alpha(\mathbf{k}')\cdot\mathbf{w}^*_\alpha(\mathbf{k}')]. \tag{94}$$

Here, $\overline{S}^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ represents the mode-to-mode energy transfer from $\mathbf{w}_\alpha(\mathbf{p})$ to $\mathbf{w}_\alpha(\mathbf{k}')$ with the mediation of $\mathbf{u}(\mathbf{q})$ (full $\mathbf{u}$); whereas $\overline{S}^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q})$ is the mode-to-mode energy transfer from $\mathbf{w}_\beta(\mathbf{p})$ to $\mathbf{w}_\alpha(\mathbf{k}')$ with the mediation of $\mathbf{u}(\mathbf{q})$ ($\beta \neq \alpha$). Also, note that

$$\overline{S}^{\alpha\alpha}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \overline{S}^{\alpha\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0, \tag{95}$$

$$\overline{S}^{\alpha\beta}(\mathbf{k}'|\mathbf{p}|\mathbf{q}) + \overline{S}^{\beta\alpha}(\mathbf{p}|\mathbf{k}'|\mathbf{q}) = 0. \tag{96}$$

These two equations are the same as Eqs. (42) and (43), except that $S \to \overline{S}$. Therefore, following the same arguments as in Sec. III, we can deduce that KE is partially conserved along these interactions. We rewrite the mode-to-mode transfer in terms of $(\mathbf{k},\mathbf{p},\mathbf{q})$ with $\mathbf{k} = \mathbf{p} + \mathbf{q}$ as

$$\overline{S}^{\alpha\alpha}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}+\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}_\alpha(\mathbf{p})\cdot\mathbf{w}^*_\alpha(\mathbf{k})\}], \tag{97}$$

$$\overline{S}^{\alpha\beta}(\mathbf{k}|\mathbf{p}|\mathbf{q}) = \tfrac{1}{2}\mathrm{Im}[\{(\mathbf{k}+\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{w}_\beta(\mathbf{p})\cdot\mathbf{w}^*_\alpha(\mathbf{k})\}]. \tag{98}$$

Note that Eqs. (97) and (98) are equivalent to mode-to-mode transfer Eqs. (46) and (47) of Sec. III. Thus, we derive formulas for the mode-to-mode KE transfers for compressible turbulence while using $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ or $\mathbf{v} = \rho\mathbf{u}$ variables.

The formulas for the energy fluxes in the two frameworks are similar, except that $S \to \overline{S}$. For example,

$$\overline{\Pi}_R(K) = \overline{\Pi}^{R<}_{R>}(K) = \sum_{k>K}\sum_{p\leqslant K}\overline{S}^{RR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{99}$$

$$\overline{\Pi}_C(K) = \overline{\Pi}^{C<}_{C>}(K) = \sum_{k>K}\sum_{p\leqslant K}\overline{S}^{CC}(\mathbf{k}|\mathbf{p}|\mathbf{q}) \tag{100}$$

are equivalent to Eqs. (53) and (54) defined in Sec. IV. Similarly, the cross fluxes,

$$\overline{\Pi}^{R<}_{C<}(K) = \sum_{k\leqslant K}\sum_{p\leqslant K}\overline{S}^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{101}$$

$$\overline{\Pi}^{R>}_{C>}(K) = \sum_{k>K}\sum_{p>K}\overline{S}^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{102}$$

$$\overline{\Pi}^{R<}_{C>}(K) = \sum_{k>K}\sum_{p\leqslant K}\overline{S}^{CR}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{103}$$

$$\overline{\Pi}^{C<}_{R>}(K) = \sum_{k>K}\sum_{p\leqslant K}\overline{S}^{RC}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{104}$$

are similar to Eqs. (55)–(58).

Hence, we expect the numerical values of the energy fluxes computed in the two frameworks to be identical. We will present detailed energy flux computations in a future communication. However, for an illustration, in Appendix C we show that the total kinetic-energy fluxes computed using $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ or $\mathbf{v} = \rho\mathbf{u}$ variables are the same.

VI. COMPARISON WITH PAST WORKS

Finally, we compare our results with those of past works. In 1990, Kida and Orszag [20] simulated compressible turbulence on 64^3 grid and computed energy transfers among the rotational and compressive velocity components and the internal energy. They primarily forced either rotation or compression component and reported cumulative energy transfers. Subsequent works by Miura and Kida [27], Graham *et al.* [35], and Schmidt and Grete [23] advanced the field by investigating multiscale spectral energy transfers in compressive turbulence. In the following, we contrast our findings with these studies, highlighting similarities and differences.

Graham *et al.* [35] investigated compressible magnetohydrodynamic turbulence using $E_u(\mathbf{k}) = \mathrm{Re}[\mathbf{v}(\mathbf{k}) \cdot \mathbf{u}^*(\mathbf{k})]/2$ as the modal kinetic energy and derived a dynamical equation for the energy transfers, which is Eq. (19) of Sec. II of our paper. Graham *et al.* [35] further decomposed the transfer term $T_u(\mathbf{k})$ of Eq. (20) as

$$T_u(\mathbf{k}) = T_{ua}(\mathbf{k}) + T_{uc}(\mathbf{k}), \tag{105}$$

where

$$T_{ua}(\mathbf{k}) = -\tfrac{1}{2}\mathrm{Re}[\mathbf{u}(\mathbf{k}) \cdot [\widehat{\mathbf{u} \cdot \nabla \mathbf{v}}]^*(\mathbf{k}) + \mathbf{v}^*(\mathbf{k}) \cdot [\widehat{\mathbf{u} \cdot \nabla \mathbf{u}}](\mathbf{k})], \tag{106}$$

$$T_{uc}(\mathbf{k}) = -\tfrac{1}{2}\mathrm{Re}[\mathbf{u}(\mathbf{k}) \cdot [\widehat{\mathbf{v}\nabla \cdot \mathbf{u}}]^*(\mathbf{k})], \tag{107}$$

with the hat symbol (e.g., $\widehat{\mathbf{u}}$) denoting the Fourier transform. They interpret $T_{ua}(\mathbf{k})$ as energy transfer due to advection, and $T_{uc}(\mathbf{k})$ as the transfer arising from compressibility effects; the latter term vanishes in the incompressible limit. These terms can be rewritten as

$$T_{ua}(\mathbf{k}) = \tfrac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{\mathbf{p} \cdot \mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p}) \cdot \mathbf{u}^*(\mathbf{k}) + \mathbf{u}(\mathbf{p}) \cdot \mathbf{v}^*(\mathbf{k})\}], \tag{108}$$

$$T_{uc}(\mathbf{k}) = \tfrac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{\mathbf{q} \cdot \mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p}) \cdot \mathbf{u}^*(\mathbf{k})\}]. \tag{109}$$

Schmidt and Grete [23] followed a similar approach to Graham *et al.* [35] but used the density-weighted velocity $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ to define the kinetic energy and its shell-to-shell spectral transfer function $\mathcal{T}_{uu}$. They derived that

$$\mathcal{T}_{ua}(\mathbf{k}) = \sum_{\mathbf{k}\in k}\sum_{\mathbf{p}\in p} \mathrm{Im}[\{\mathbf{p} \cdot \mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p}) \cdot \mathbf{w}^*(\mathbf{k})\}], \tag{110}$$

$$\mathcal{T}_{uc}(\mathbf{k}) = \tfrac{1}{2}\sum_{\mathbf{k}\in k}\sum_{\mathbf{p}\in p} \mathrm{Im}[\{\mathbf{q} \cdot \mathbf{u}(\mathbf{q})\}\{\mathbf{w}(\mathbf{p}) \cdot \mathbf{w}^*(\mathbf{k})\}]. \tag{111}$$

The sum of $T_{ua}(\mathbf{k})$ and $T_{uc}(\mathbf{k})$ or $\mathcal{T}_{ua}(\mathbf{k})$ and $\mathcal{T}_{uc}(\mathbf{k})$, yields the kinetic-energy flux. But the advective and compressive terms of Graham *et al.* [35] and Schmidt and Grete [23] fail to clearly distinguish energy transfers between rotational and compressive components. In contrast, our formalism achieves this distinction.

In the present paper, we go beyond Eq. (19) and derive mode-to-mode energy transfer, $S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q})$, which is kinetic-energy transfer from wavenumber $\mathbf{p}$ to wavenumber $\mathbf{k}$ with mediation of wavenumber $\mathbf{q}$. In addition, we separate the velocity field into rotational and compressional components, i.e., $\mathbf{u}(\mathbf{k}) = \mathbf{u}_R(\mathbf{k}) + \mathbf{u}_C(\mathbf{k})$, after which we derived mode-to-mode energy transfers for these components. These transfers enable us to clearly derive energy transfers T_{RR} (from R to R), T_{CC} (C to C), and T_{CR} (R to C), and then energy fluxes among the rotational and compressive velocity components and the internal energy (see Figs. 1 and 4). Thus, our framework goes beyond the works of Graham *et al.* [35] and Schmidt and Grete [23] in a significant way.

To contrast our transfers terms with past works, we expand Graham *et al.*'s [35] T_{ua} and T_{uc} using the rotational and compressive components as

$$T_{ua}(\mathbf{k}) = \frac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_R(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k}) + \mathbf{v}_R(\mathbf{p})\cdot\mathbf{u}_C^*(\mathbf{k}) + \mathbf{v}_C(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k}) + \mathbf{v}_C(\mathbf{p})\cdot\mathbf{u}_C^*(\mathbf{k})\}]$$
$$+ \frac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}_R(\mathbf{p})\cdot\mathbf{v}_R^*(\mathbf{k}) + \mathbf{u}_R(\mathbf{p})\cdot\mathbf{v}_C^*(\mathbf{k}) + \mathbf{u}_C(\mathbf{p})\cdot\mathbf{v}_R^*(\mathbf{k}) + \mathbf{u}_C(\mathbf{p})\cdot\mathbf{v}_C^*(\mathbf{k})\}], \tag{112}$$

$$T_{uc}(\mathbf{k}) = \frac{1}{2}\sum_{\mathbf{p}} \mathrm{Im}[\{\mathbf{q}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}_R(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k}) + \mathbf{v}_R(\mathbf{p})\cdot\mathbf{u}_C^*(\mathbf{k}) + \mathbf{v}_C(\mathbf{p})\cdot\mathbf{u}_R^*(\mathbf{k}) + \mathbf{v}_C(\mathbf{p})\cdot\mathbf{u}_C^*(\mathbf{k})\}]. \tag{113}$$

Therefore, both T_{ua} and T_{uc} involve T_{RR}, T_{CC}, T_{RC}, and T_{CR}. Hence, they do not separate the energy transfers between rotational and compressive components. We overcome this difficulty in our framework using mode-to-mode energy transfers. Note, however, that our formulas for pressure dilatation and viscous dissipation are similar to those of Graham *et al.* [35] and Schmidt and Grete [23].

Aluie [38] introduced a coarse-graining framework to analyze the kinetic-energy cascade, which was later extended by Wang *et al.* [33] to separately examine the cascades of rotational and compressible kinetic energy. Banerjee and Galtier [40] derived an exact relation for the two-point correlation function for compressible polytropic turbulence under the assumptions of statistical homogeneity. While these approaches have provided important insights into the nature of kinetic-energy cascades in compressible turbulence, they do not fully resolve the detailed scale-to-scale energy transfers.

The above discussion shows that our paper advances the energy transfers in compressible turbulence in a significant way.

VII. CONCLUSIONS

Despite extensive past research, a comprehensive understanding of compressible turbulence remains elusive, and a consistent description of energy fluxes has been lacking. This paper presents a significant advancement by introducing a novel framework for computing energy transfers and fluxes in compressible flows. Specifically, we derive the mode-to-mode energy-transfer rates in Fourier space and demonstrate detailed energy conservation within triads of interacting modes for the nonlinear terms. The formalism captures both kinetic and internal energy exchanges, thereby extending the detailed energetics of incompressible turbulence analysis to the compressible regime.

We further decompose the kinetic energy into rotational and compressive components and derive their respective mode-to-mode transfer terms, including the cross-interaction contributions. Leveraging these results, we construct analytical expressions for rotational and compressive kinetic-energy fluxes, cross-transfer fluxes between them, and transfers from kinetic to internal energy via pressure and viscous dissipation. This framework enables a detailed quantification of energy exchanges between rotational, compressive, and internal energy components. Interestingly, several formulas for the computation of energy transfers in incompressible magnetohydrodynamic flows resemble those derived in this paper, thus demonstrating the universality of the mode-to-mode transfer formalism.

Our work advances the study of compressible turbulence by providing a precise framework for analyzing energy fluxes, interscale energy transfers, and locality, which will be presented in another communication. Past works report various energy fluxes in magnetoconvection. The mode-to-mode energy transfers presented in this paper will complement the above work with more detailed computation, e.g., energy fluxes for the rotational and compressive velocity components. Thus, our work sheds important new light on the dynamics of compressible turbulence and opens avenues for exploring more complex flows, such as compressible convection and magnetohydrodynamics in planetary, stellar, and galactic environments, as well as in engineering flows. The equations for the compressible turbulence have similarities with quantum turbulence, where we propose to extend the present energy-transfer formalism.

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DATA AVAILABILITY

The data that support the findings of this article are openly available [54], embargo periods may apply.

APPENDIX A: DERIVATION OF DYNAMICAL EQUATION FOR $E_u(\mathbf{k})$ AND COMBINED ENERGY TRANSFERS

The momentum and velocity equations for a compressible flow, expressed in nondimensional tensorial form, are given by

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij}\sigma - \tau_{ij}) = \rho F_i, \tag{A1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_j}(\delta_{ij}\sigma - \tau_{ij}) = F_i. \tag{A2}$$

Using a new field $\mathbf{v} = \rho\mathbf{u}$ and an application of the Fourier transform to Eqs. (A1) and (A2) yield

$$\frac{d}{dt}\mathbf{v}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{v}(\mathbf{p}) - i\mathbf{k}\sigma(\mathbf{k}) - \mathbf{d}(\mathbf{k}) + \mathbf{F}'(\mathbf{k}), \tag{A3}$$

$$\frac{d}{dt}\mathbf{u}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\mathbf{u}(\mathbf{p}) - \tilde{\sigma}(\mathbf{k}) - \tilde{\mathbf{d}}(\mathbf{k}) + \mathbf{F}(\mathbf{k}), \tag{A4}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$ and

$$\tilde{\sigma} = \frac{\nabla\sigma}{\rho}, \quad \mathbf{d} = -\partial_j\tau_{ij}, \quad \tilde{\mathbf{d}} = \frac{\mathbf{d}}{\rho}, \quad \mathbf{F}' = \rho\mathbf{F}.$$

Taking the dot product of Eq. (A3) with $\mathbf{u}^*(\mathbf{k})$ and Eq. (A4) with $\mathbf{v}^*(\mathbf{k})$ yields

$$\mathbf{u}^*(\mathbf{k})\cdot\frac{d}{dt}\mathbf{v}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\} - i\sigma(\mathbf{k})\{\mathbf{k}\cdot\mathbf{u}^*(\mathbf{k})\}$$
$$-\mathbf{d}(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k}) + \mathbf{F}'(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k}), \tag{A5}$$

$$\mathbf{v}^*(\mathbf{k})\cdot\frac{d}{dt}\mathbf{u}(\mathbf{k}) = -i\sum_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}^*(\mathbf{k})\} - \tilde{\sigma}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})$$
$$-\tilde{\mathbf{d}}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k}) + \mathbf{F}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k}). \tag{A6}$$

By adding Eqs. (A5) and (A6) we obtain

$$\begin{aligned}
&\mathbf{u}^*(\mathbf{k})\cdot\frac{d}{dt}\mathbf{v}(\mathbf{k})+\mathbf{v}^*(\mathbf{k})\cdot\frac{d}{dt}\mathbf{u}(\mathbf{k})\\
&= -i\sum_{\mathbf{p}}\{\mathbf{k}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}^*(\mathbf{k})\}-i\sum_{\mathbf{p}}\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}^*(\mathbf{k})\}\\
&\quad-[i\sigma(\mathbf{k})\{\mathbf{k}\cdot\mathbf{u}^*(\mathbf{k})\}+\tilde{\sigma}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})]-[\mathbf{d}(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})+\tilde{\mathbf{d}}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})]\\
&\quad+[\mathbf{F}'(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})+\mathbf{F}(\mathbf{k})\cdot\mathbf{v}^*(\mathbf{k})].
\end{aligned}\tag{A7}$$

We further add Eq. (A7) with its complex conjugate and obtain [see Eq. (19)]

$$\frac{d}{dt}E_u(\mathbf{k})=T_u(\mathbf{k})-Q_{I,u}(\mathbf{k})-D_{I,u}(\mathbf{k})+\mathcal{F}_u(\mathbf{k}).\tag{A8}$$

The terms on the right-hand side represent the nonlinear transfer, pressure dilatation, viscous dissipation, and energy injection by $\mathbf{F}$, respectively; these terms are defined in Eqs. (20)–(23) of the main text.

For a single triad $(\mathbf{k}',\mathbf{p},\mathbf{q})$ with $\mathbf{k}'+\mathbf{p}+\mathbf{q}=0$ [11,31], the energy equation is

$$\frac{d}{dt}E_u(\mathbf{k}')=S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})-Q_{I,u}(\mathbf{k}')-D_{I,u}(\mathbf{k}')+\mathcal{F}_u(\mathbf{k}'),\tag{A9}$$

where

$$S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})=-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}(\mathbf{k}')\}-\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}(\mathbf{k}')\}]+\mathbf{p}\leftrightarrow\mathbf{q}\tag{A10}$$

is the combined kinetic energy transfer to wavenumber $\mathbf{k}'$ from wavenumbers $\mathbf{p}$ and $\mathbf{q}$. Here $\mathbf{p}\leftrightarrow\mathbf{q}$ represents the energy transfer to wavenumber $\mathbf{k}'$ from wavenumber $\mathbf{q}$ with $\mathbf{p}$ acting as mediator. This is the second term in Eq. (25). Similarly, we can write the combined kinetic energy transfers to wavenumbers $\mathbf{p}$ and $\mathbf{q}$ as

$$S^{uu}(\mathbf{p}|\mathbf{k}',\mathbf{q})=-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})\}-\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{k}')\cdot\mathbf{v}(\mathbf{p})\}]+\mathbf{k}'\leftrightarrow\mathbf{q},\tag{A11}$$

$$S^{uu}(\mathbf{q}|\mathbf{p},\mathbf{k}')=-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{q}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}-\{\mathbf{p}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}(\mathbf{q})\}]+\mathbf{p}\leftrightarrow\mathbf{k}'.\tag{A12}$$

An addition of Eqs. (A10)–(A12) yields

$$\begin{aligned}
&S^{uu}(\mathbf{k}'|\mathbf{p},\mathbf{q})+S^{uu}(\mathbf{p}|\mathbf{k}',\mathbf{q})+S^{uu}(\mathbf{q}|\mathbf{p},\mathbf{k})\\
&= -\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}(\mathbf{k}')\}-\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}(\mathbf{k}')\}]\\
&\quad-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{v}(\mathbf{q})\cdot\mathbf{u}(\mathbf{k}')\}-\{\mathbf{q}\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{u}(\mathbf{q})\cdot\mathbf{v}(\mathbf{k}')\}]\\
&\quad-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{v}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{p})\}-\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{q})\}\{\mathbf{u}(\mathbf{k}')\cdot\mathbf{v}(\mathbf{p})\}]\\
&\quad-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{p}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{v}(\mathbf{q})\cdot\mathbf{u}(\mathbf{p})\}-\{\mathbf{q}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{u}(\mathbf{q})\cdot\mathbf{v}(\mathbf{p})\}]\\
&\quad-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{q}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{v}(\mathbf{p})\cdot\mathbf{u}(\mathbf{q})\}-\{\mathbf{p}\cdot\mathbf{u}(\mathbf{k}')\}\{\mathbf{u}(\mathbf{p})\cdot\mathbf{v}(\mathbf{q})\}]\\
&\quad-\tfrac{1}{2}\mathrm{Im}[\{\mathbf{q}\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{v}(\mathbf{k}')\cdot\mathbf{u}(\mathbf{q})\}-\{\mathbf{k}'\cdot\mathbf{u}(\mathbf{p})\}\{\mathbf{u}(\mathbf{k}')\cdot\mathbf{v}(\mathbf{q})\}]]\\
&= 0,
\end{aligned}\tag{A13}$$

which is the statement of detailed conservation law for compressible hydrodynamics.

APPENDIX B: SHELL-TO-SHELL ENERGY TRANSFERS IN COMPRESSIBLE TURBULENCE

For a preliminary test of the locality in compressible turbulence, we perform direct numerical simulations (DNS) of compressible turbulence by solving Eqs. (1)–(3) using the GPU-enabled Python code Dhara [51]. It employs the computationally efficient MacCormack scheme, which is

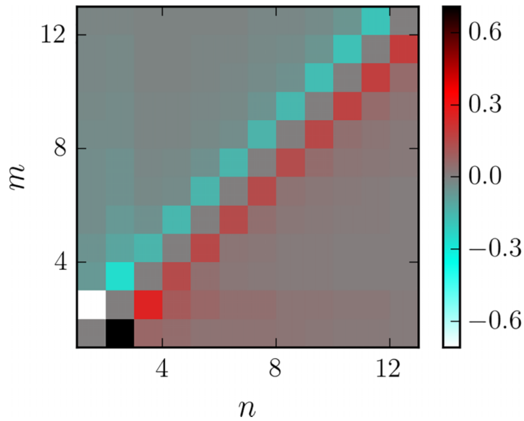

FIG. 5. For $M_t = 0.45$, the normalized shell-to-shell transfer $\mathcal{T}_n^m/\Pi_u$, where Π_u is the maximum value of the total kinetic-energy flux.

second-order accurate in space and time [52,53]. The domain is a $(2\pi)^3$ periodic box with a uniform grid of 1024^3 points. We set the Prandtl number $\mathrm{Pr} = 1$, the reference Mach number $M_0 = 1$, and the ratio of specific heats $C_p/C_v = \gamma = 1.001$. The near-unity value of γ ensures the flow is nearly isothermal. We initiate our simulation with a stationary fluid ($\mathbf{u} = 0$) and uniform density and temperature ($\rho = 1, T = 1$). We employ Kida and Orszag's [20] scheme to force the velocity field at large scales. The amplitudes of the rotational and compressive forcing components were set to yield an injection ratio of $\epsilon_R^{\mathrm{inj}}/\epsilon_C^{\mathrm{inj}} \approx 2$, maintaining a turbulent Mach number of $M_t = 0.45$.

Using the numerical data, we compute the shell-to-shell transfer from shell m to shell n

$$\mathcal{T}_n^m = \sum_{\mathbf{k}\in s_n} \sum_{\mathbf{p}\in s_m} S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}). \tag{B1}$$

In this Appendix, we compute the energy transfer for the full velocity field, not for the rotational or compressive component. The first two wave-number shells have $k \in (0, 2]$ and $k \in (2, 4]$. For $n > 2$, the wave-number shells are (k_n, k_{n+1}), with $k_n = 2^{(n+6)/4}$ [44]. We choose maximum $n = 14$, ensuring that our calculations include the inertial range ($2 < m, n \leqslant 9$) and a fraction of the dissipation range ($m, n > 9$).

Figure 5 illustrates $\mathcal{T}_n^m/\Pi_u$, where Π_u is the maximum value of the total kinetic-energy flux in the inertial range. Note that $\Pi_u = \Pi_C + \Pi_R + \Pi_{C>}^{R<} + \Pi_{R>}^{C<}$. As shown in the figure, in the inertial range, the maximal energy transfer is from shell m to $m + 1$. Thus, we demonstrate forward and local kinetic-energy transfer in compressible turbulence for a moderate Mach number of 0.45. In the future, we will compute detailed shell-to-shell energy transfers for the rotational and compressive components, as well as for the internal energy.

APPENDIX C: ENERGY FLUX COMPARISON BETWEEN THE TRANSFORMATIONS $\mathbf{v} = \rho\mathbf{u}$ AND $\mathbf{w} = \sqrt{\rho}\mathbf{u}$

To validate the equivalence of the two formulations described in Secs. III and V, we perform numerical simulations with $M_t = 0.45$ (see Appendix B for simulation details). Using the numerical data, we compute the total kinetic-energy fluxes:

$$\Pi_u(K) = \sum_{k>K} \sum_{p\leqslant K} S^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{C1}$$

$$\overline{\Pi}_u(K) = \sum_{k>K} \sum_{p\leqslant K} \overline{S}^{uu}(\mathbf{k}|\mathbf{p}|\mathbf{q}), \tag{C2}$$

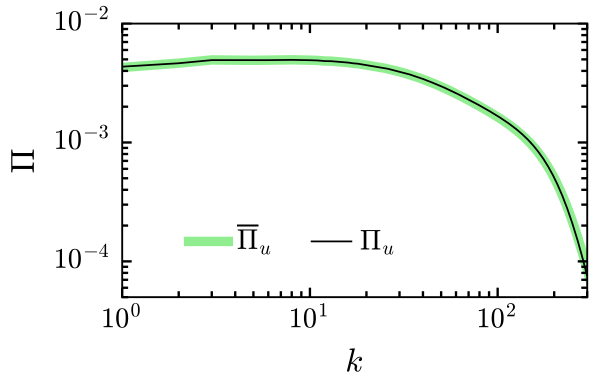

FIG. 6. For $M_t = 0.45$, the total kinetic-energy fluxes $\Pi_u(K)$ (black curve) and $\overline{\Pi}_u(K)$ (green curve) computed using Eqs. (28) and (84) for $\mathbf{v} = \rho\mathbf{u}$ and $\mathbf{w} = \sqrt{\rho}\mathbf{u}$, respectively.

using the mode-to-mode transfer formulas defined in Eqs. (28) and (84) for the formalisms with $\mathbf{v} = \rho\mathbf{u}$ and $\mathbf{w} = \sqrt{\rho}\mathbf{u}$, respectively. We plot these fluxes in Fig. 6 and observe them to be nearly the same across all wavenumbers. Thus, we show equivalence of the energy flux formulas with $\mathbf{v} = \rho\mathbf{u}$ and $\mathbf{w} = \sqrt{\rho}\mathbf{u}$ variables. In the future, we plan to perform further tests to compare the two formalisms.

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