

Renormalization Group analysis of Navier-Stokes equation

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I. Governing Equations:

- Incompressible Navier-Stokes equation in Fourier space is:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) = -ik_i p(\mathbf{k}, t) - ik_j \int \frac{d\mathbf{p}}{(2\pi)^d} u_j(\mathbf{k} - \mathbf{p}, t) u_i(\mathbf{p}, t)$$

- Pressure can be found out by taking divergence of above:

$$p(\mathbf{k}) = -\frac{k_i k_j}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^d} [u_j(\mathbf{k} - \mathbf{p}, t) u_i(\mathbf{p}, t)] .$$

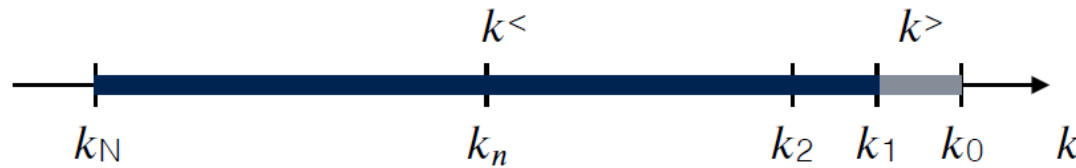
- Combining using the tensor notation, Verma (2005):

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) = -\frac{i}{2} P_{ijm}^+(\mathbf{k}) \int \frac{d\mathbf{p}}{(2\pi)^d} [u_j(\mathbf{p}, t) u_m(\mathbf{k} - \mathbf{p}, t)]$$

where, $P_{ijm}^+(\mathbf{k}) = k_j P_{im}(\mathbf{k}) + k_m P_{ij}(\mathbf{k}); \quad P_{im}(\mathbf{k}) = \delta_{im} - \frac{k_i k_m}{k^2};$

II. Renormalization Procedure:

- **Step 1:** Divide (k_N, k_0) wavenumbers into N shells in inertial range.



Taken from Verma (2005).

$$k_n = h^n k_0$$

- **Step 2:** We will coarse-grain (k_1, k_0) shell in first step and then come down to lower shells recursively.

$$\langle u_i^>(\hat{k}) \rangle = 0$$

$$\langle u_i^<(\hat{k}) \rangle = u_i^<(\hat{k})$$

- **Step 3:** Equation for $u_i^<(\vec{k}, t)$:

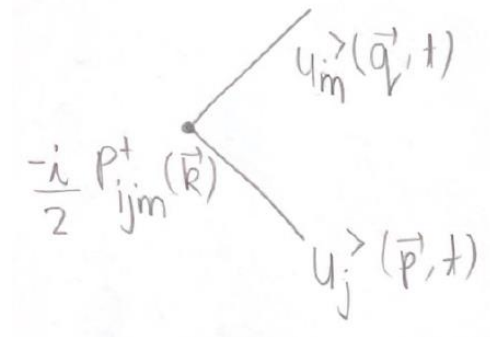
$$(\partial_t + \nu_{(0)} k^2) u_i^<(\vec{k}, t) = -\frac{i}{2} P_{ijm}^+(\vec{k}) \int \frac{d\vec{p}}{(2\pi)^d} [u_j^<(\vec{p}, t) u_m^<(\vec{q}, t) + 2u_j^<(\vec{p}, t) u_m^>(\vec{q}, t) + u_j^>(\vec{p}, t) u_m^>(\vec{q}, t)]$$

- After ensemble average,

$$(\partial_t + \nu_{(0)} k^2) u_i^<(\vec{k}, t) = -\frac{i}{2} P_{ijm}^+(\vec{k}) \int \frac{d\vec{p}}{(2\pi)^d} [u_j^<(\vec{p}, t) u_m^<(\vec{q}, t) + \langle u_j^>(\vec{p}, t) u_m^>(\vec{q}, t) \rangle]$$

$$I = -\frac{i}{2} P_{ijm}^+(\vec{k}) \int \frac{d\vec{p}}{(2\pi)^d} \langle u_j^>(\vec{p}, t) u_m^>(\vec{q}, t) \rangle$$

- We must find the above integral which we will expand using Green's function to first order.



A handwritten diagram illustrating a vertex in a Feynman diagram. The vertex is represented by a central point with the label $-\frac{i}{2} P_{ijm}^+(\vec{k})$ next to it. Two lines extend from this vertex: one line goes upwards and to the right, labeled $u_m^>(\vec{q}, t)$, and the other line goes downwards and to the right, labeled $u_j^>(\vec{p}, t)$.

- **Step 4:** We expand $u_j^>(\vec{p}, t)$ using Green's function:

$$u_j^>(\vec{p}, t) = \int d\vec{s} dt' G(\vec{p}, t-t') \left[-i p_{jim}^+(\vec{p}) u_m^>(\vec{s}, t') - i p_{ijm}^+(\vec{k}) C_{jj}(\vec{p}, t-t') u_i^<(\vec{k}, t') \right]$$

Diagram illustrating the expansion of $u_j^>(\vec{p}, t)$ using Green's function $G(\vec{p}, t-t')$. The expansion involves two terms: a direct term $-i p_{jim}^+(\vec{p}) u_m^>(\vec{s}, t')$ and a loop term $-i p_{ijm}^+(\vec{k}) C_{jj}(\vec{p}, t-t') u_i^<(\vec{k}, t')$.

- Thus, we can expand $u_m^>(\vec{q}, t)$ similarly and:

$$-i p_{ijm}^+(\vec{k}) \left[u_m^>(\vec{q}, t) + u_j^>(\vec{p}, t) \right] = -i p_{ijm}^+(\vec{k}) \left[C_{mm}(\vec{q}, t-t') u_m^>(\vec{q}, t-t') + G(\vec{p}, t-t') u_i^<(\vec{k}, t') \right] + \dots$$

Diagram illustrating the expansion of $u_m^>(\vec{q}, t)$ and $u_j^>(\vec{p}, t)$ using Green's functions $C_{mm}(\vec{q}, t-t')$ and $G(\vec{p}, t-t')$. The expansion involves two terms: a direct term $-i p_{ijm}^+(\vec{k}) u_m^>(\vec{q}, t)$ and a loop term $-i p_{ijm}^+(\vec{k}) C_{jj}(\vec{p}, t-t') u_i^<(\vec{k}, t')$.

- **Step 5:** Using Markovian approximation, we can take out $u_i^<(\vec{k}, t')$ out of time integral:

$$I = -\frac{(d-1)}{2} u_i^<(\vec{k}, t) P_{ijm}^+(\vec{k}) \int_0^{t'} dt' \int^\Delta \frac{d\vec{p}}{(2\pi)^d} [P_{jim}^+(\vec{p}) G(\vec{p}, t-t') C_{mm}(\vec{q}, t-t') \\ + P_{mij}^+(\vec{q}) G(\vec{q}, t-t') C_{jj}(\vec{p}, t-t')]$$

- Taking $G(\vec{p}, t-t') = \theta(t-t') e^{-\nu_{(0)}(p)p^2(t-t')}$ & $C_{jj}(\vec{p}, t-t') = (d-1)C(p) e^{-\nu_{(0)}(p)p^2(t-t')}$

$$\boxed{\delta\nu_{(0)}(k) = \frac{(d-1)^2}{2k^2} P_{ijm}^+(\vec{k}) \int^\Delta \frac{d\vec{p}}{(2\pi)^d} \frac{P_{jim}^+(\vec{p})C(q) + P_{mij}^+(\vec{q})C(p)}{\nu_{(0)}(p)p^2 + \nu_{(0)}(q)q^2}}$$

where,

$$P_{ijm}^+(\vec{k}) P_{jim}^+(\vec{p}) = \vec{k} \cdot \vec{p} (d-5) + 4 \frac{(\vec{k} \cdot \vec{p})^3}{k^2 p^2}$$

$$P_{ijm}^+(\vec{k}) P_{mij}^+(\vec{q}) = \vec{k} \cdot \vec{q} (d-5) + 4 \frac{(\vec{k} \cdot \vec{q})^3}{k^2 q^2}$$

- **Step 6:** Thus, the renormalized viscosity after first shell integral is:

$$\nu_{(1)}(k) = \nu_{(0)}(k) + \delta\nu_{(0)}(k);$$

- And, the general $(n + 1)$ shell renormalized viscosity is:

$$\nu_{(n+1)}(k) = \nu_{(n)}(k) + \delta\nu_{(n)}(k)$$

- **Step 7:** Using self-consistent theory, we put:

$$C(k) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E(k) \qquad E(k) = K_{Ko} \Pi^{2/3} k^{-5/3},$$

$$\nu_{(n)}(k_n k') = (K_{Ko})^{1/2} \Pi^{1/3} k_n^{-4/3} \nu_{(n)}^*(k')$$

- We reiterate the coarse graining till $\nu_{(m+1)}^*(k') \approx \nu_{(m)}^*(k')$ i.e. they converge for large n .

Thank You!