Renormalization Group analysis of Navier-Stokes equation

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I. Governing Equations:

- Incompressible Navier-Stokes equation in Fourier space is:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) \;=\; -ik_i p\,(\mathbf{k}, t) - ik_j \int \frac{d\mathbf{p}}{(2\pi)^d} u_j(\mathbf{k}-\mathbf{p}, t) u_i(\mathbf{p}, t)$$

- Pressure can be found out by taking divergence of above:

$$p\,(\mathbf{k}) = -\frac{k_i k_j}{k^2} \int \frac{d\mathbf{p}}{(2\pi)^d} \left[u_j(\mathbf{k}-\mathbf{p}, t) u_i(\mathbf{p}, t)\right].$$

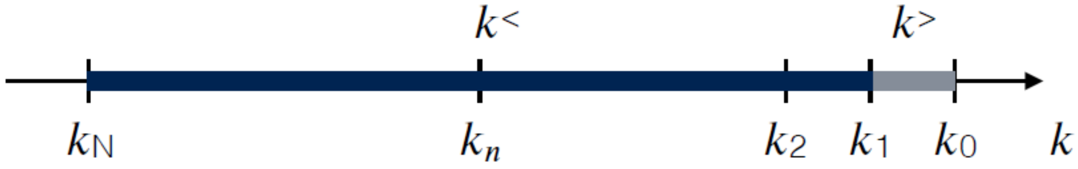
- Combining using the tensor notation, Verma (2005):

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_i(\mathbf{k}, t) \;=\; -\frac{i}{2} P^{+}_{ijm}(\mathbf{k}) \int \frac{d\mathbf{p}}{(2\pi)^d} \left[u_j(\mathbf{p}, t) u_m(\mathbf{k}-\mathbf{p}, t)\right]$$

where, $$P^{+}_{ijm}(\mathbf{k}) \;=\; k_j P_{im}(\mathbf{k}) + k_m P_{ij}(\mathbf{k}); \quad P_{im}(\mathbf{k}) \;=\; \delta_{im} - \frac{k_i k_m}{k^2};$$

II. Renormalization Procedure:

- **Step 1**: Divide (k_N, k_0) wavenumbers into N shells in inertial range.



Taken from Verma (2005).

$$k_n = h^n k_0$$

- **Step 2:** We will coarse-grain (k_1, k_0) shell in first step and then come down to lower shells recursively.

$$\left\langle u_i^>(\hat{k}) \right\rangle = 0$$
$$\left\langle u_i^<(\hat{k}) \right\rangle = u_i^<(\hat{k})$$

- **Step 3:** Equation for $u_i^{<}(\vec{k},t)$:

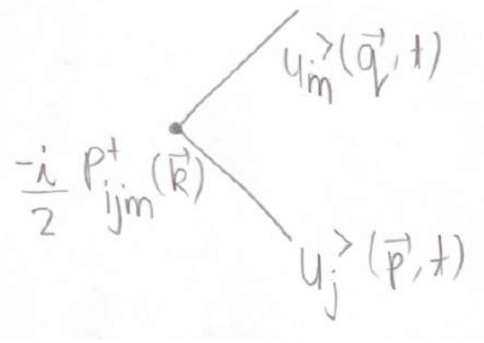
$$(\partial_t + \nu_{(0)} k^2) u_i^{<}(\vec{k},t) = -\frac{i}{2} P^{+}_{ijm}(\vec{k}) \int \frac{d\vec{p}}{(2\pi)^d} [u_j^{<}(\vec{p},t) u_m^{<}(\vec{q},t) + 2u_j^{<}(\vec{p},t) u_m^{>}(\vec{q},t) + u_j^{>}(\vec{p},t) u_m^{>}(\vec{q},t)]$$

- After ensemble average,

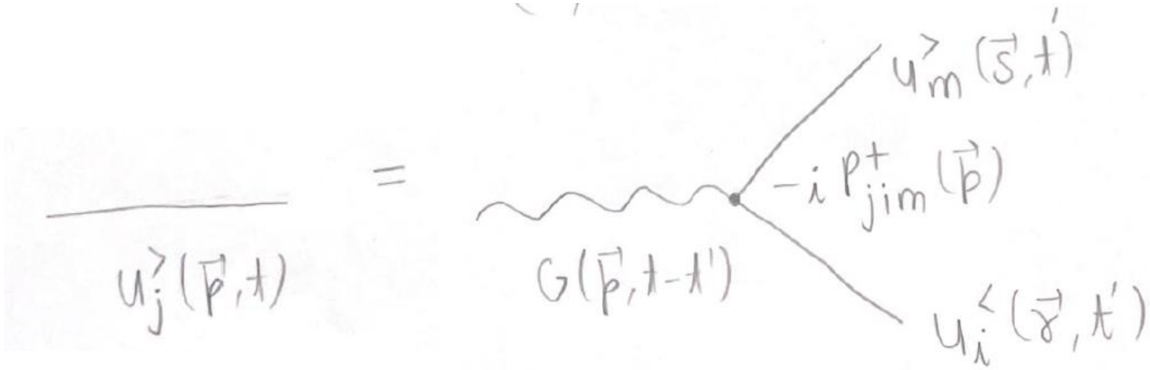
$$(\partial_t + \nu_{(0)} k^2) u_i^{<}(\vec{k},t) = -\frac{i}{2} P^{+}_{ijm}(\vec{k}) \int \frac{d\vec{p}}{(2\pi)^d} [u_j^{<}(\vec{p},t) u_m^{<}(\vec{q},t) + < u_j^{>}(\vec{p},t) u_m^{>}(\vec{q},t) >]$$

$$I = -\frac{i}{2} P^{+}_{ijm}(\vec{k}) \int^{\Delta} \frac{d\vec{p}}{(2\pi)^d} < u_j^{>}(\vec{p},t) u_m^{>}(\vec{q},t) >$$

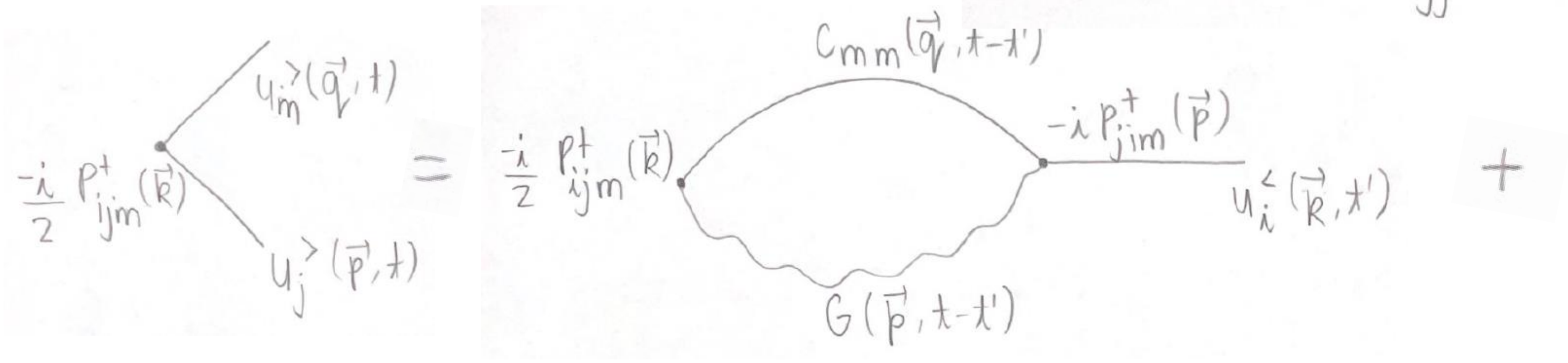
- We must find the above integral which we will expand using Green's function to first order.

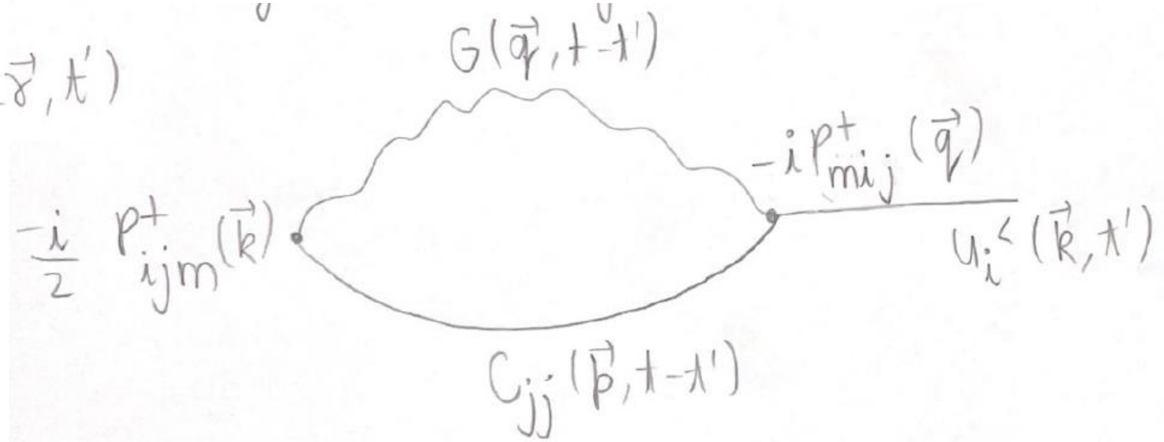


- **Step 4:** We expand $u_j^{>}(\vec{p},t)$ using Green's function:



- Thus, we can expand $u_m^{>}(\vec{q},t)$ similarly and:





- **Step 5:** Using Markovian approximation, we can take out $u_i^{<}(\vec{k}, t')$ out of time integral:

$$I = -\frac{(d-1)}{2} u_i^{<}(\vec{k}, t) P_{ijm}^{+}(\vec{k}) \int_0^{t'} dt' \int^{\Delta} \frac{d\vec{p}}{(2\pi)^d} [P_{jim}^{+}(\vec{p}) G(\vec{p}, t-t') C_{mm}(\vec{q}, t-t') + P_{mij}^{+}(\vec{q}) G(\vec{q}, t-t') C_{jj}(\vec{p}, t-t')]$$

- Taking $G(\vec{p}, t-t') = \theta(t-t') e^{-\nu_{(0)}(p) p^2 (t-t')}$ & $C_{jj}(\vec{p}, t-t') = (d-1) C(p) e^{-\nu_{(0)}(p) p^2 (t-t')}$

$$\boxed{\delta\nu_{(0)}(k) = \frac{(d-1)^2}{2k^2} P_{ijm}^{+}(\vec{k}) \int^{\Delta} \frac{d\vec{p}}{(2\pi)^d} \frac{P_{jim}^{+}(\vec{p}) C(q) + P_{mij}^{+}(\vec{q}) C(p)}{\nu_{(0)}(p) p^2 + \nu_{(0)}(q) q^2}}$$

where,

$$P_{ijm}^{+}(\vec{k}) P_{jim}^{+}(\vec{p}) = \vec{k} \cdot \vec{p} (d-5) + 4 \frac{(\vec{k} \cdot \vec{p})^3}{k^2 p^2}$$

$$P_{ijm}^{+}(\vec{k}) P_{mij}^{+}(\vec{q}) = \vec{k} \cdot \vec{q} (d-5) + 4 \frac{(\vec{k} \cdot \vec{q})^3}{k^2 q^2}$$

- **Step 6:** Thus, the renormalized viscosity after first shell integral is:

$$\nu_{(1)}(k) \ = \ \nu_{(0)}(k) + \delta\nu_{(0)}(k);$$

- And, the general $(n + 1)$ shell renormalized viscosity is:

$$\nu_{(n+1)}(k) = \nu_{(n)}(k) + \delta\nu_{(n)}(k)$$

- **Step 7:** Using self-consistent theory, we put:

$$C(k) = \frac{2(2\pi)^d}{S_d(d-1)} k^{-(d-1)} E(k) \qquad E(k) \ = \ K_{Ko}\Pi^{2/3}k^{-5/3},$$

$$\nu_{(n)}(k_n k') = (K_{Ko})^{1/2}\Pi^{1/3}k_n^{-4/3}\nu^*_{(n)}(k')$$

- We reiterate the coarse graining till $\nu^*_{(m+1)}(k') \approx \nu^*_{(m)}(k')$ i.e. they converge for large n.

Thank You!